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අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2021(2022)
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2021 (2022)
General Certificate of Education (Adv. Level) Examination, 2021 (2022)

උසස් ගණිතය I
உயர் கணிதம் I
Higher Mathematics I

11 E I

පැය තුනයි
மூன்று மணித்தியாலம்
Three hours

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மேலதிக வாசிப்பு நேரம் - 10 நிமிடங்கள்
Additional Reading Time - 10 minutes

Use additional reading time to go through the question paper, select the questions you will answer and decide which of them you will prioritise.

Index Number							

Instructions:

※ *This question paper consists of two parts;*
Part A (Questions **1** - **10**) and **Part B** (Questions **11** - **17**).

※ **Part A:**
*Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*

※ **Part B:**
*Answer **five** questions only. Write your answers on the sheets provided.*

※ *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*

※ *You are permitted to remove **only** **Part B** of the question paper from the Examination Hall.*

For Examiners' Use only

(11) Higher Mathematics I		
Part	**Question No.**	**Marks**
A	1	
	2	
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B	11	
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	17	
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In Numbers	
In Words	

Code Numbers

Marking Examiner		
Checked by:	1	
	2	
Supervised by:		

[see page two

Part A

1. Factorize: $x^4(y^2 - z^2) + y^4(z^2 - x^2) + z^4(x^2 - y^2)$.

2. Let a relation R be defined on the set of all integers $\mathbb{Z}$ by aRb if $5a + b$ is divisible by 3. Show that R is an equivalence relation on $\mathbb{Z}$ and write down the equivalence class of 0.

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3. Let $f(x)=\frac{x}{x-3}$ for $x\neq 3$ and $g^{-1}(x)=2x-1$ for $x\in\mathbb{R}$.

Find $f^{-1}(x)$ and $g(x)$, and show that $g(2f^{-1}(0))=\frac{1}{2}$.

4. Show that

$$\begin{vmatrix} x^3+x & x+1 & x-2 \\ 2x^3+3x+1 & 3x & 3x-3 \\ x^3+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = x\begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}.$$

[see page four

5. The chord joining the points $P \equiv (ap^2, 2ap)$ and $Q \equiv (aq^2, 2aq)$ on the parabola $y^2 = 4ax$ passes through the focus of the parabola. Show that $pq = -1$ and **deduce** that the tangents drawn at P and Q to the parabola are perpendicular.

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6. Let $f(x) = \begin{cases} \dfrac{\alpha x + |x|}{\beta x - |x|} & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$ and $g(x) = \begin{cases} \dfrac{\sqrt{1+x} - 1}{\alpha x} & \text{if } x \neq 0 \\ -\beta & \text{if } x = 0, \end{cases}$

where $\alpha > 0$ and $\beta \in \mathbb{R}$.

It is given that $f(x)$ and $g(x)$ are continuous at $x = 0$. Find the values of α and β.

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[see page five

7. Let $f(x)=\begin{cases} x^3, & \text{if} \quad x\geq 0, \\ -x^2, & \text{if} \quad -1<x<0, \\ -x-2, & \text{if} \quad x\leq -1. \end{cases}$

Show that $f(x)$ is differentiable at $x=0$ and non-differentiable at $x=-1$.
Write down $f'(x)$ for $x\neq -1$.

8. Solve the differential equation $\tan y\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{1}{1+x}+(1+x)e^x\sec y=0$, subject to the condition $y=1$ when $x=0$.

[see page six

9. Let f be a continuous real-valued function on $\mathbb{R}$ and let $a > 0$.

Show that $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(a-x)+f(a+x)\}\,dx$

10. Sketch the curve whose polar equation is given by $r = 2\cos\theta + 4\sin\theta$.

Find the polar equation of the tangent to the above curve at the point with polar coordinates $\left(4, \frac{\pi}{2}\right)$ on it.

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கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2021(2022)
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උසස් ගණිතය I
உயர் கணிதம் I
Higher Mathematics I

11 E I

Part B

※ Answer **five** questions only.

11. (*a*) Let A, B and C be subsets of a universal set S. Stating clearly the Laws of Algebra of sets that you use, show that

(i) $(A-B)\cup(A-C)=A-(B\cap C)$,

(ii) $(A'-B)\cap C'=(A'-C)-(B-C)$,

where $A-B$ is defined by $A\cap B'$.

(*b*) In a group of Football, Basketball and Volleyball players, it is given that

(i) 8 players can play Football and Basketball,

(ii) 5 players can play Football and Volleyball,

(iii) 7 players can play Basketball and Volleyball,

(iv) 29 players can play Football or Basketball,

(v) 30 players can play Football or Volleyball and

(vi) 25 players can play Basketball or Volleyball

Find how many players can play Football.

12. (*a*) Let $a, b, c > 0$.

(i) Show that $ab \le \frac{1}{2}(a^2+b^2)$ and **deduce** that $abc^2 \le \frac{1}{4}\left(a^4+b^4+2c^4\right)$.

(ii) **Hence**, show that $abc \le \left(\dfrac{a^4+b^4+c^4}{a+b+c}\right)$. Show that the equality holds if and only if $a=b=c$.

(*b*) The transformation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 6 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ maps points in the xy-plane to the points in the $x'y'$-plane. Find the equations of the straight lines which are mapped onto them-selves. Find the image of the line $y=2x-1$ in the $x'y'$-plane.

[see page eight

13. State and prove **De Moivre's Theorem** for a positive integral index.

Using **De Moivre's Theorem**, show that

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ and

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

Hence, show that $\cot 4\theta = \dfrac{\cot^4\theta - 6\cot^2\theta + 1}{4(\cot^3\theta - \cot\theta)}$.

Solve $\cot 4\theta = \sqrt{3}$ and show that $x = \cot\left(\frac{\pi}{24}\right)$ is a solution of the equation $x^4 - 4\sqrt{3}x^3 - 6x^2 + 4\sqrt{3}x + 1 = 0$.

Write down the other solutions of this equation also in the form $\cot\left(\frac{k\pi}{24}\right)$ stating the values of k.

Deduce that $\cot\frac{\pi}{24} + \cot\frac{7\pi}{24} + \cot\frac{13\pi}{24} + \cot\frac{19\pi}{24} = 4\sqrt{3}$.

14.(*a*) Let C_1 and C_2 be the curves given by $y = (x-1)^2 + 1$ and $(y-2)^2 = 16x$ respectively. Sketch the graphs of C_1 and C_2 in the same diagram indicating the coordinates of their points of intersection.

Find the area of the region R bounded by the curves C_1 and C_2.

Also, find the volume of the solid generated by rotating the region R through 2π radians about the line $y = 1$.

(*b*) A family of curves is defined by the differential equation $\dfrac{dy}{dx} = \dfrac{2x - y + 5}{-x + 2y + 5}$.

Find the equation of the curve of the family that passes through the origin.

15.(*a*) Let $I_n = \displaystyle\int_0^1 x^n(1-x)^{\frac{3}{2}}\,dx$, for $n \in \mathbb{Z}^+$.

Show that, $I_n = \left(\dfrac{2n}{2n+5}\right)I_{n-1}$ for $n \geq 2$.

Hence, find the value of $\displaystyle\int_0^1 x^4(1-x)^{\frac{3}{2}}\,dx$.

(*b*) Find the Maclaurin series of $\cos x$ and e^x in ascending powers of x up to and including the term in x^3.

Hence, obtain the Maclaurin series of $e^{-x}\cos(x^2)$ in ascending powers of x up to and including the term in x^3.

Using this, find an approximate value for $\displaystyle\int_0^{0.1} xe^{-x}\cos(x^2)\,dx$.

[see page nine

16. Show that the equation of the tangent at the point $(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $bx\cos\theta + ay\sin\theta = ab$ and the equation of the tangent at $(r\cos\phi, r\sin\phi)$ to the circle $x^2+y^2=r^2$ is $x\cos\phi + y\sin\phi = r$.

Let C be the circle $x^2+y^2=36$ and S be the ellipse $\frac{x^2}{9^2}+\frac{y^2}{4^2}=1$.

Suppose that $(6\cos\phi, 6\sin\phi)$ is a point of intersection of the circle C and the ellipse S. Show that $\tan^2\phi = \frac{4}{9}$.

Hence, or otherwise show that the acute angle between the tangents to the ellipse S and the circle C at the points of intersection is $\tan^{-1}\left(\frac{5}{9}\right)$.

17. (*a*) Let $f(x) = \dfrac{\cos^2 x}{2+2\sin x\cos x+\sin^2 x}$ for $x \in \mathbb{R}$.

(i) Show that $0 \le f(x) \le \frac{3}{5}$ for $x \in \mathbb{R}$.

(ii) Solve the equations $f(x) = \frac{3}{5}$ and $f(x) = 0$, and sketch the graph of $y = f(x)$ for $0 \le x \le \frac{\pi}{2}$.

(*b*) The following table gives values of the function $f(x)$ correct to two decimal places for values of x between 0 to 1.2 at intervals of length 0.2.

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2
$f(x)$	1.12	2.01	0.00	1.11	1.65	2.42	1.61

Using Simpson's Rule, find an approximate value for $I = \int_0^{1.2} f(x)\,\mathrm{d}x$.

Hence, find an approximate value for $\int_0^{1.2} x f'(x)\,\mathrm{d}x$.

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கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2021(2022)
General Certificate of Education (Adv. Level) Examination, 2021(2022)

උසස් ගණිතය II
உயர் கணிதம் II
Higher Mathematics II

11 E II

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மூன்று மணித்தியாலம்
Three hours

අමතර කියවීම් කාලය - මිනිත්තු 10 යි
மேலதிக வாசிப்பு நேரம் - 10 நிமிடங்கள்
Additional Reading Time - 10 minutes

Use additional reading time to go through the question paper, select the questions you will answer and decide which of them you will prioritise.

Index Number | | | | | | | |

Instructions:

* *This question paper consists of two parts;*
  Part A (Questions **1 - 10**) and **Part B** (Questions **11 - 17**).
* **Part A:**
  *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
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* *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
* *You are permitted to remove **only*** **Part B** *of the question paper from the Examination Hall.*

* *Statistical Tables will be provided.*
* *g denotes the acceleration due to gravity.*

For Examiners' Use only

(11) Higher Mathematics II		
Part	**Question No.**	**Marks**
A	1	
	2	
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B	11	
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Marking Examiner		
Checked by:	1	
	2	
Supervised by:		

[see page two

Part A

1. Let $A \equiv (-2, -1, -1)$, $B \equiv (3, 1, 2)$ and $C \equiv (1, \alpha, -\beta)$, where $\alpha, \beta > 0$, be three points such that the angle between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{2\pi}{3}$ and $|\overrightarrow{OA}| = |\overrightarrow{OC}|$. Find the values of α and β.

2. Two forces, $\mathbf{F}_1 = 3\mathbf{i} + \alpha\mathbf{j} + \mathbf{k}$ and $\mathbf{F}_2 = \mathbf{i} + \beta\mathbf{j} + \mathbf{k}$ act at the points with position vectors $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\gamma\mathbf{i} + 8\mathbf{k}$ respectively where $\alpha, \beta, \gamma \in \mathbb{R}$. It is given that their lines of action pass through the point $4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$. Find the values of α, β and γ. Write down the equation of the line of action of their resultant.

[see page three

3. A uniform solid right circular cone of radius a and height $4a$ and of density σ floats partially immersed in a homogeneous liquid of density ρ with its vertex at a distance a above the free surface of the liquid. Find the value of the ratio $\frac{\sigma}{\rho}$.

Find the smallest weight of the particle that can be attached to the vertex of the cone to make it totally immersed in the liquid.

4. The acceleration of a particle P at time t is given by $a(t) = 6t\,\mathbf{i} - \cos t\,\mathbf{j} + e^{t}\mathbf{k}$. The position vector and the velocity of the particle P at $t = 0$ are $\mathbf{j} + \mathbf{k}$ and $\mathbf{k}$ respectively. Find the position vector of P at time t.

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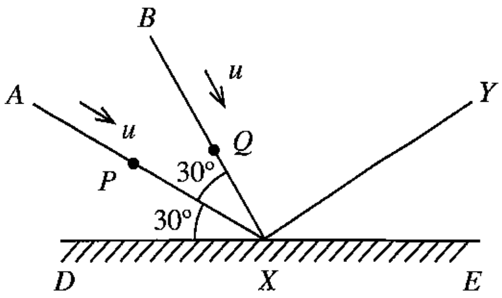
[see page four



5. Two particles P and Q each of mass m are moving on a smooth horizontal plane, each with speed u, towards on a smooth vertical wall which meets the wall in the line DE. They both hit the wall a the point X with Q arriving at X before P. The direction of motion of P is along AX, where $A\hat{X}D = 30°$. The direction of motion of Q is along BX, where $B\hat{X}D = 60°$. After rebounding from the wall both P and Q move in the same direction XY (See the figure). The coefficient of restitution between P and the wall is e. Show that the coefficient of restitution between Q and the wall is $\frac{e}{3}$.

Show that after rebounding from the wall, the speeds of P and Q are in the ratio $\sqrt{3}:1$.

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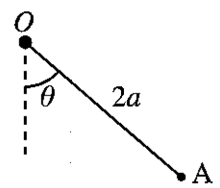

6. A uniform rod OA of mass m and length $2a$ is free to rotate in a vertical plane about the end O which is fixed. It is held in a position making an angle $\frac{2\pi}{3}$ with the downward vertical and then released. Show that when the rod makes an angle θ with the downward vertical the angular speed $\dot{\theta}$ satisfies $a\dot{\theta}^2 = \frac{3}{4}g(1+2\cos\theta)$.

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[see page five

7. In a restaurant, the probability that a randomly selected customer orders a cup of tea with his breakfast is 0.6. If a random sample of 5 customers is selected,
find the probability that,

(i) exactly one customer,

(ii) less than 3 customers,

order tea with their breakfast.

8. In an urban area, power cuts occur randomly twice a month. Find the probability that in a given month, there will be

(i) no power cuts,

(ii) at least 2 power cuts.

[see page six

9. The probability density function $f(x)$ of a continuous random variable X is given by

$$f(x)=\begin{cases} \frac{1}{c}x^2 & , \quad |x|\leq 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

where c is a constant. Find the value of c.
Also find $E(X)$ and $V(X)$.

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10. An unbiased cubic die is rolled once. Let X be the value obtained and let $Y=\frac{1}{2}X^2$.
Find $E(X)$ and $E(Y)$.

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கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2021(2022)
General Certificate of Education (Adv. Level) Examination, 2021(2022)

උසස් ගණිතය II
உயர் கணிதம் II
Higher Mathematics II

11 E II

Part B

* Answer **five** questions only.

11. Let $A \equiv (1, 0, 0)$, $B \equiv (0, 1, 0)$ and $C \equiv (0, 0, 1)$ be three points. Forces, $2\overrightarrow{AB}$, $3\overrightarrow{AC}$ and $\overrightarrow{BC}$ act along AB, AC and BC in the sense indicated by the letters respectively.

(i) Show that the system, reduces to a single force $\mathbf{R}$ acting through the point A and a couple $\mathbf{G}$, where $\mathbf{R}$ and $\mathbf{G}$ are to be determined.

(ii) Now a force $\mathbf{F}$ is introduced to the above system.

(*a*) If $\mathbf{F}$ acts through the origin and the system reduces to a couple, find $\mathbf{F}$ and the magnitude of the couple.

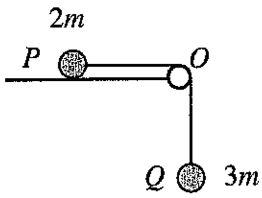
(*b*) If $\mathbf{F}$ acts at the point with the position vector $\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ and the system is in equilibrium, find the values of c and d.

12. A circular lamina of radius a is immersed with its surface vertical in a homogeneous liquid of constant density ρ such that its centre O at a depth a below the free surface of the liquid. Show that the magnitude of the liquid thrust on the lamina is $\pi a^3 \rho g$, and that the centre of pressure of the lamina lies on the vertical diameter, at a distance $\frac{a}{4}$ below the centre O.

A solid right circular cone of radius a and height $2a$ is immersed in a homogeneous liquid of constant density ρ, with its highest point on the free surface of the liquid and its axis horizontal. Find the magnitude, direction and the line of action of the thrust on the curved surface of the cone.

13. A particle P of mass $2m$ placed on a rough horizontal table is connected to a particle Q of mass $3m$ by a light inextensible string which passes over a small smooth pulley O fixed at the edge of the table. Particles P and Q and the string all lie in a vertical plane.

Let the coefficient of friction between P and the table be $\frac{1}{2}$. The system is released from rest with the string taut, as shown in the diagram. Show that the particles start to move.



The particle Q moves in a resistive medium that offers a resistance mkv when its speed is v, where $k (> 0)$ is a constant. Show that $5\frac{dv}{dt} = 2g - kv$.

Find the time taken by P to reach a speed of $\frac{g}{k}$ and the distance travelled by P during this period. (Assume that the length of the string is long enough for P not to reach O during this period.)

[*see page eight*

14. Two smooth spheres, A and B, of equal radius, lie on a smooth horizontal table. A is of mass m and B is of mass $2m$. The spheres are projected towards each other and when they colide, the line joining their centers is parallel to $\mathbf{j}$ and velocities of A and B are $2\mathbf{i} + 3\mathbf{j}$ and $-\mathbf{i} + \alpha\mathbf{j}$ respectively, where $\alpha > 0$. The coefficient of restitution between A and B is $\frac{1}{2}$ and the velocity of the sphere Q just after the collision is $-\mathbf{i} + 3\mathbf{j}$. Find

(i) the value of α

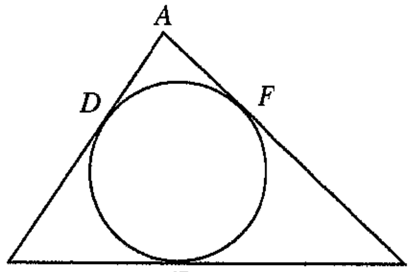
(ii) the velocity of the sphere P just after the collision

(iii) the loss in kinetic energy caused by the collision

(iv) the impulse on A from B.

15. A frame consists of three uniform rods, each of mass m and length $2a$, rigidly joined together to form the triangle ABC, together with a uniform circular ring of mass m inscribed in the triangle, as shown in the figure. The ring is rigidly fixed to the rods AB, BC and CA at D, E and F respectively, where D, E and F are mid-points of AB, BC and CA respectively.

Show that the moment of inertia of the frame about the axis through A perpendicular to the plane of the frame is $\frac{23}{3}ma^2$.



The frame can rotate about a fixed smooth horizontal axis through A, the axis being perpendicular to the plane of the frame. The frame is given a small displacement from the equilibrium position in which the center of mass of the frame is below A, and released from rest. Show that the motion of the frame is approximately simple harmonic and its period is $2\pi\sqrt{\frac{23a}{8\sqrt{3}g}}$.

16. (*a*) Suppose the discrete random variable X has the probability distribution given below:

X	0	1	2	3	4
$P(X=x)$	0.1	0.3	0.4	0.15	0.05

Let $Y = 2X + 1$. The probability distribution of Y is given by the following table.

Y	1	3	5	7	9
$P(Y=y)$	0.1	0.3	p	q	0.05

(i) Find the values of p and q.

(ii) Find $E(Y)$ and $\text{Var}(Y)$.

(iii) Find $P(Y > 3)$, and **hence** find $P(X > 1)$.

[*see page nine*

(*b*) (i) Let X be the number of tails obtained when 3 unbiased coins are tossed. Find the probability distribution of X and **hence** find $E(X)$ and $\text{Var}(X)$.

(ii) If the value of X is an odd number, a biased cubical die with the probability of getting 3 or 6 on the face, equals to $\frac{2}{3}$ is rolled. Otherwise, another biased cubical die with the probability of getting 3 or 6 on the face, equals to $\frac{1}{3}$ is rolled. Variable Y is defined as follows:

$$Y = \begin{cases} 2, & \text{if the value on the face of the die is divisible by 3} \\ 1, & \text{otherwise} \end{cases}$$

Find the probability distribution of Y and **hence**, find $E(Y)$ and $\text{Var}(Y)$.

17. (*a*) The probability density function of the continuous random variable X is given by

$$f_X(x) = \begin{cases} 10x^2(1-x) & , \quad 0 < x < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find $E(X)$ and $\text{Var}(X)$.

Also, find $P\left(X < \frac{1}{2}\right)$.

If the continuous random variable Y is defined by $Y = \frac{3X+2}{4}$, find $E(Y)$ and $\text{Var}(Y)$.

(*b*) Suppose that daily household expenditures on transport in an urban area are normally distributed with a mean of Rs. 2000 and a standard deviation of Rs. 400.

(i) Find the probability that the daily transport cost of a randomly selected household exceeds Rs. 2500.

(ii) It is given that the daily transport cost of 10% of the households exceeds Rs. k. Find the value of k.

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