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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2016 අගෝස්තු
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2016 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2016

උසස් ගණිතය I
உயர் கணிதம் I
Higher Mathematics I

11 E I

පැය තුනයි
மூன்று மணித்தியாலம்
Three hours

Index Number							

Instructions:

* *This question paper consists of two parts;* **Part A** (Questions 1 - 10) and **Part B** (Questions 11 - 17).
* **Part A:** *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
* **Part B:** *Answer **five** questions only. Write your answers on the sheets provided.*
* *At the end of the time allotted, tie the answer scripts of the two parts together so that **Part A** is on top of **Part B** and hand them over to the supervisor.*
* *You are permitted to remove **only** **Part B** of the question paper from the Examination Hall.*

For Examiners' use only

(11) Higher Mathematics I

Part	Question No.	Marks
A	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	
B	11	
	12	
	13	
	14	
	15	
	16	
	17	
	Total	
	Percentage	

Paper I	
Paper II	
Total	
Final Marks	

Final Marks

In Numbers	
In Words	

Code Numbers

Marking Examiner		
Checked by:	1	
	2	
Supervised by:		

[see page two

Part A

1. Factorize: $(x+y)^3(x-y)+(y+z)^3(y-z)+(z+x)^3(z-x)$.

2. A relation R is defined on $\mathbb{R}$ by xRy if $x^2-y^2-x+y=0$. Show that R is an equivalence relation on $\mathbb{R}$.

[see page three

3. Let $f(x) = (ax+1)^{\frac{1}{3}}$ and $g(x) = 3x + 4$ for $x \in \mathbb{R}$ be such that $(f \circ g)(1) = 2$, where a is a real constant. Find the value of a.

Let $h(x) = (f \circ f)(x)$. Find $h^{-1}(x)$.

4. Show that $\begin{vmatrix} b+c & c & b \\ c & a+c & a \\ b & a & a+b \end{vmatrix} = 4abc$.

[see page four

5. Show that the the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is $y + tx = 2at + at^3$.

Find the value of t if this normal passes through the focus of the parabola $y^2 = 4ax$.

6. Let $a \in \mathbb{R}$, $b \geq 1$, and $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x\tan^{-1}\left(\frac{1}{x}\right), & \text{if } x < 0, \\ a, & \text{if } x = 0, \\ \sqrt{b-1+x}, & \text{if } x > 0. \end{cases}$$

If f is continuous at $x = 0$, find the values of a and b.

[see page five

7. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x, & \text{if } x < 1 \\ x^2 - 2x + 2, & \text{if } x \geq 1 \end{cases}$$

Is f differentiable at $x = 1$? Justify your answer.

Write down $f'(x)$ for all $x \neq 1$.

8. Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$, subject to the condition $y = \ln 2$ when $x = 0$.

[see page six

9. Let f and g be real-valued continuous functions on the interval [0, 1] satisfying the equation $2x\, f(x^2) + 3g(x) = 14\,x$ for $x \in [0, 1]$.

Show that if $\int_0^1 f(x)\,dx = 1$, then $\int_0^1 g(x)\,dx = 2$.

10. Sketch the curves whose polar equations are given by $r = 2$ and $r = 2\,(\cos\theta - \sin\theta)$ in the same diagram, and find the polar coordinates of their points of intersection.

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கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2016 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2016

උසස් ගණිතය I
உயர் கணிதம் I
Higher Mathematics I

11 E I

PART B

※ Answer **five** questions only.

11. (*a*) Let A, B and C be subsets of a universal set S. Stating clearly the Laws of Algebra of sets that you use, show that

(i) $(B - A) \cup (C - A) = (B \cup C) - A$ and

(ii) $A \cap (B - C) = (A \cap B) - (A \cap C)$,

where the set $A - B$ is defined by $A - B = A \cap B'$.

(*b*) In a survey involving of 40 people who had visited at least one of the three cities A, B and C, it was revealed that 22 people had been to city A, 23 to city B and 19 to city C. It was also revealed that 18 people had been to the two cities A and B, 11 to A and C, 13 to B and C and 11 to all three cities. Find the number of people who

(i) had been to A or B,

(ii) had been to B and C, but not to A,

(iii) had not been to B or C.

12. (*a*) Let a, b and c be positive real numbers. Assuming the inequality $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$, prove that

(i) $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$,

(ii) $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{9}{2}$ and

(iii) $(1-a)(1+a)^2 \leq \frac{32}{27}$ for $0 < a < 1$.

(*b*) The transformation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ in the xy- plane maps the point $(a, 2)$ to the point (a, b), where a and b are real constants. Find the values of a and b.

This transformation maps the line $x = 1$ to a line $px' + qy' + r = 0$, where p, q and r are real constants. Find the values of p, q and r.

Find the equation of the line in the xy- plane which gets mapped onto the line $2x' + y' = 1$ under this transformation.

[see page eight

13. State and prove **De Moivre's theorem** for a positive integral index.

Let $\omega_k = \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)$ for $k = 0, 1, 2, \ldots$ Show that $\omega_k^5 = 1$ for $k = 0, 1, 2, \ldots$ and **hence** write down the five distinct roots of the equation $z^5 - 1 = 0$.

Deduce that $\omega_1, \omega_2, \omega_3$ and ω_4 are the four distinct roots of the equation $z^4 + z^3 + z^2 + z + 1 = 0$.

Deduce further that $z^4 + z^3 + z^2 + z + 1 = \left\{z^2 - 2\cos\left(\frac{2\pi}{5}\right) z + 1\right\}\left\{z^2 - 2\cos\left(\frac{4\pi}{5}\right) z + 1\right\}$.

By comparing the coefficients of z^2 and z^3 of this identity, obtain the quadratic equation with integer coefficients having $\cos\left(\frac{2\pi}{5}\right)$ and $\cos\left(\frac{4\pi}{5}\right)$ as its roots.

Hence show that $\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5} - 1}{4}$.

14. (*a*) Let C_1 and C_2 be the curves given by $x = y^2$ and $x = 2 - y^2$ respectively. Sketch graphs of C_1 and C_2 in the same diagram, indicating the coordinates of their points of intersection.

Find the area of the region S bounded by the two curves C_1 and C_2.

Also find the volume of the solid generated by rotating the region S through four right angles about the line $x = 3$.

(*b*) A family of curves satisfies the differential equation $\frac{dy}{dx} = \frac{4x + y}{x - 4y}$. By substituting $y = xV$ show that the given differential equation gets transformed to $\frac{1 - 4V}{4(1 + V^2)}\, dV = \frac{1}{x}\, dx$.

Hence show that the family of curves has the Cartesian representation $\frac{1}{2}\tan^{-1}\left(\frac{y}{x}\right) - \ln\left(x^2 + y^2\right) = \lambda$, where $\lambda \in \mathbb{R}$.

Also, obtain the differential equation satisfied by the orthogonal trajectories of this family of curves.

15. (*a*) Let $I_n = \int_0^1 (1 - x^3)^n x\, dx$ for $n \in \mathbb{Z}^+$.

Show that $(3n + 2)I_n = 3nI_{n-1}$ for $n = 2, 3, \ldots$ and **deduce** that $I_n = \frac{3^n n!}{(3n+2)(3n-1)\cdots 8\cdot 5\cdot 2}$

for $n \in \mathbb{Z}^+$.

(*b*) Let $y = e^{\cos x}$ for $x \in \mathbb{R}$. Show that $\frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} + (\cos x)\, y = 0$.

Obtain the Maclaurin series expansion of y up to and including the term of x^4.

Hence find an approximate value for the integral $\int_0^1 e^{\cos x}\, dx$.

[see page nine

16. Verify that the point $T \equiv \left(\frac{a}{2}\left(t+\frac{1}{t}\right), \frac{b}{2}\left(t-\frac{1}{t}\right)\right)$, where t ($\neq 0$) is a parameter, lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Obtain the equation of the tangent to the hyperbola at the point T.

Show that the points $P \equiv (at, bt)$ and $Q \equiv (at', -bt')$ lie on the asymptotes of the hyperbola.

It is given that the mid-point R of PQ lies on the hyperbola. Show that $tt' = 1$ and that the line PQ is tangent to the hyperbola at R.

Show also that $OP \cdot OQ = a^2 + b^2$, where O is the origin.

Let L and M be the points at which the perpendiculars drawn from R to the asymptotes of the hyperbola meet the asymptotes. Show further that $RL \cdot RM = \frac{a^2 b^2}{a^2 + b^2}$.

17. (*a*) Let $f(x) = \frac{3 + \sin^2 x}{2 + \cos^2 x}$ for $x \in \mathbb{R}$.

(i) Show that $1 \leq f(x) \leq 2$ for $x \in \mathbb{R}$.

(ii) Solve the equations $f(x) = 1$ and $f(x) = 2$.

(iii) Sketch the graph of $y = f(x)$ for $0 \leq x \leq \pi$.

(*b*) The following table gives the values of the function $f(x) = \ln(1 + x^2)$ correct to three decimal places, for the values of x indicated there:

x	0	0.5	1.0	1.5	2.0
$f(x)$	0	0.223	0.693	1.179	1.909

Using Simpson's Rule with the values given in the above table, find an approximate value for the area bounded by the curves $y = \ln(1 + x^2)$, $x = 0$, $x = 2$ and $y = 0$.

Deduce an approximate value for $\int_{-2}^{2} \ln\sqrt{1 + x^2}\, dx$.

* * *

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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2016 අගෝස්තු
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2016 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2016

උසස් ගණිතය II
உயர் கணிதம் II
Higher Mathematics II

11 E II

පැය තුනයි
மூன்று மணித்தியாலம்
Three hours

Index Number							

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- *You are permitted to remove **only*** **Part B** *of the question paper from the Examination Hall.*
- *Statistical tables will be provided.*
- g denotes the acceleration due to gravity.

For Examiners' Use only

(11) Higher Mathematics II		
Part	**Question No.**	**Marks**
A	1	
	2	
	3	
	4	
	5	
	6	
	7	
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B	11	
	12	
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	15	
	16	
	17	
	Total	
	Percentage	

Paper **I**	
Paper **II**	
Total	
Final Marks	

Final Marks

In Numbers	
In Words	

Code Numbers

Marking Examiner	
Checked by: 1	
2	
Supervised by:	

[see page two

Part A

1. The position vector $\mathbf{r}$ of a point P, with respect to a fixed origin O, is given by $\mathbf{r} = \left(\frac{1}{2}a\sin\theta\right)\mathbf{i} + \left(\frac{\sqrt{3}}{2}a\sin\theta\right)\mathbf{j} + (a\cos\theta)\mathbf{k}$, where $\theta\ (0 \le \theta \le \pi)$ is a scalar parameter and a is a positive constant.

   Show that (i) $\mathbf{r}\cdot\mathbf{r} = a^2$ (ii) $\mathbf{r}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta} = 0$ and (iii) $\mathbf{r}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta} = \frac{a^2}{2}\left(-\sqrt{3}\,\mathbf{i} + \mathbf{j}\right)$.

2. Through the points A, B and C with position vectors $a\mathbf{i}$, $b\mathbf{j}$ and $c\mathbf{k}$, respectively, three forces $P(b\mathbf{j} + c\mathbf{k})$, $P(c\mathbf{k} + a\mathbf{i})$ and $P(a\mathbf{i} + b\mathbf{j})$ act, where P is a positive constant and a, b, c are constants such that $abc \neq 0$. Show that this system reduces to a single resultant force whose line of action is $\mathbf{r} = \lambda(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$, where λ is a parameter. Find the magnitude of the resultant force.

[see page three

3. Triangle ABC is the cross-section through the centre of gravity, perpendicular to the three parallel edges, of a uniform triangular prism. The prism floats freely in a homogeneous liquid with the edge through A on the free surface, the edge through B below the free surface and the edge through C above the free surface. By considering the lines of action of its weight and the upward liquid thrust, show that the face of the prism containing BC is vertical.

4. The position vector $\mathbf{r}$ at time t, of a particle of mass m with respect to a fixed origin is given by $\mathbf{r} = a[(\sin 2\omega t)\mathbf{i} + (1 - \cos 2\omega t)\mathbf{j}]$, where a and ω are positive constants. Show that

(i) its path is a circle with centre C with position vector $a\mathbf{j}$ and radius a,

(ii) its angular momentum about C is $2ma^2\omega\mathbf{k}$, and

(iii) $\ddot{\mathbf{r}} + 4\omega^2(\mathbf{r} - a\mathbf{j}) = \mathbf{0}$.

[see page four

5. A smooth sphere of mass m moving with uniform velocity $\mathbf{u} = u(\mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha)$, where $0 < \alpha < \frac{\pi}{2}$ and u is a positive constant, on a smooth horizontal floor strikes a smooth sphere of the same radius and of mass M, initially at rest. If the velocities of m and M after impact are $v\mathbf{j}$ and $w\mathbf{i}$ respectively, show that

(i) their mutual impulse is of magnitude $mu\cos\alpha$,

(ii) the coefficient of restitution is $\frac{m}{M}$.

6. A uniform circular hoop of mass M and radius a rolls, without slipping, on a rough horizontal floor. Its plane remains vertical and the speed v of its centre is constant. Find the kinetic energy of the hoop.

[see page five

7. A random variable R takes integer values $r = 1, 2, 3, \cdots, n$, each with probability $\frac{1}{n}$. Find $E(R)$, the expected value of R.

Assuming the formula $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, also find the variance of R.

8. A discrete random variable X takes values –2, 0, 2 with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ respectively.

Obtain the probability distribution of the random variable $Y = X_1 + X_2$, where X_1 and X_2 are two independent observations of X. Show that the standard deviation of Y is 2.

[see page six

9. The probability density function $f(x)$, of a continuous random variable X is given by

$$f(x) = \begin{cases} kx(1-x) & \text{for } 0 \le x \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(i) Show that $k = 6$

(ii) Find $E(X)$ and $E(X^2)$.

10. A random variable X is uniformly distributed on the interval [1, 4].

(i) Find $P(2 \le X \le 3)$ and $P(X \le 2)$.

(ii) Find the value of a such that $P(X \ge a) = 0.6$.

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கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2016 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2016

උසස් ගණිතය II
உயர் கணிதம் II
Higher Mathematics II

11 E II

PART B

* Answer **five** questions only.

11. A system of forces consists of six forces acting along the lines specified below.

Line	Force
$\overrightarrow{OA} = \mathbf{j} + \mathbf{k}$	$P(\mathbf{j} + \mathbf{k})$
$\overrightarrow{OB} = \mathbf{k} + \mathbf{i}$	$P(\mathbf{k} + \mathbf{i})$
$\overrightarrow{OC} = \mathbf{i} + \mathbf{j}$	$P(\mathbf{i} + \mathbf{j})$
$\overrightarrow{BC} = \mathbf{j} - \mathbf{k}$	$Q(\mathbf{j} - \mathbf{k})$
$\overrightarrow{CA} = \mathbf{k} - \mathbf{i}$	$Q(\mathbf{k} - \mathbf{i})$
$\overrightarrow{AB} = \mathbf{i} - \mathbf{j}$	$Q(\mathbf{i} - \mathbf{j})$

Here P and Q are constants. Reduce the system to a single force $\mathbf{R}$ at the origin O and a couple of moment vector $\mathbf{G}$. Obtain the conditions for the system to be equivalent to

(i) a single resultant force,

(ii) a couple.

When it is given that both P and Q are non-zero, show that the system is equivalent to a wrench of pitch $\dfrac{Q}{2P}$. Find the vector equation of the central axis of the wrench and verify that it passes through the centroid of the tetrahedron $OABC$.

12. A lamina $ABCD$ in the shape of a square with $AB = a$ is completely immersed vertically in a homogeneous liquid so that the side AB lies on the free surface of the liquid. The point E is taken on the side CD such that $CE = x$ and the thrust on the trapezium $ABCE$ is equal to the thrust on the triangle ADE. Show that $x = \dfrac{a}{4}$.

Using integration, find the distance from AB to the centre of pressure of

(i) the square $ABCD$,

(ii) the triangle ADE.

By taking moments about AB, find the distance from AB to the centre of pressure of the trapezium $ABCE$.

[see page eight

13. A particle of mass m is projected up along a line of greatest slope of a smooth plane of inclination α to the horizontal, with initial speed u. The air resistance to its motion is mkv, where k is a positive constant and v is the speed. Show that the maximum distance L moved by the particle, up the plane, is given by $L = \frac{u}{k} + \frac{g}{k^2}\ln\left(\frac{g\sin\alpha}{ku + g\sin\alpha}\right)$ and find the time taken by the particle to move the distance L.

Obtain an equation relating the above initial speed u with the speed V with which the particle returns to the starting point, provided that the same resistance acts in the downward motion along the line of greatest slope.

14. A smooth sphere A moving on a smooth horizontal table impinges on an equal smooth sphere B lying at rest on the table. The direction of velocity of A makes an angle $\theta (< \frac{\pi}{4})$, with the line of centres of the spheres at the moment of impact. The coefficient of restitution between the two spheres is e $(0 < e < 1)$. Show that the magnitude J of the mutual impulse between the spheres is given by $J = \frac{1}{2}mu(1+e)\cos\theta$; where m is the mass of each sphere and u is the speed of A before the collision.

Using this impulse, or otherwise, show that the fraction δ of original kinetic energy lost due to impact is given by $\delta = \frac{1}{2}(1-e^2)\cos^2\theta$.

Show further that the tangent T of the angle of deflection of the path of A due to the collision is given by $\frac{1+e}{T} = 2t + \frac{1-e}{t}$, where $t = \tan\theta$.

Hence, show that the deflection takes a maximum value when $t = \sqrt{\frac{1-e}{2}}$ and that, in this case, $\delta = \frac{1-e^2}{3-e}$.

15. Show that the moment of inertia of a uniform rod AB of mass m and length $2a$ about an axis through its mid-point G perpendicular to AB is $\frac{1}{3}ma^2$.

A small smooth light ring is attached to the end A of the rod AB and the ring is free to move along a smooth straight wire fixed horizontally. The rod is held along and under the wire and released from rest in that position. Show that

(i) the mid point G of the rod moves in a vertical straight line L, and

(ii) the angular speed $\dot{\theta}$ of the rod when it is inclined at an angle θ to the horizontal is given by $a\dot{\theta}^2 = \frac{6g\sin\theta}{1 + 3\cos^2\theta}$.

Find the velocity of G when the rod becomes vertical.

Now, the ring breaks at the instant when the rod becomes vertical, and the rod begins to move under gravity alone. Show that in the subsequent motion of the rod the mid point G moves along the same vertical straight line L with constant acceleration g, and that the rod rotates about G with constant angular speed $\sqrt{\frac{6g}{a}}$.

[see page nine

16. (*a*) In a certain game, a player has to roll a ball **twice** down an inclined plane, and each time the ball settles in one of the five slots marked 1, 2, 4, 2, 1 separately. The probability of ball settling in any slot is $\frac{1}{5}$.

Let X = "Sum of the two marks assigned to the slots where the ball settle in." The probability distribution table for X is given below:

x	2	3	4	5	6	8
$P(X=x)$	$4p$	q	$4p$	$4p$	$4p$	p

Find the values of p and q.

Find $E(X)$ and $E(X^2)$, and show that $Var(X) = 2.4$.

(*b*) For a discrete random variable Y, the cumulative distribution function $F(y)$ is given by $F(y) = ky^2$, $y = 1, 2, 3$. Find the value of k and the probability distribution of Y.

Also, find values of $E(Y)$ and $E(3Y - 2)$.

17. (*a*) A continuous random variable X has probability density function $f(x) = \begin{cases} \frac{1}{2}(2-x), & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$

(i) Find $P(X > 1)$ and $P\left(X > 1 \middle| X > \frac{1}{2}\right)$,

(ii) Show that $E(X) = \frac{2}{3}$ and find $Var(X)$.

(*b*) The weights of packets of sugar are normally distributed with mean 500 g and standard deviation of 10 g.

(i) Find the probability that a randomly chosen packet has a weight between 490 g and 505 g.

(ii) Find the value of k such that 95% of all packets have weights between $(500 - k)$ g and $(500 + k)$ g.

(iii) Five packets are randomly chosen. Find the probability that at most two of these packets have a weight less than 495 g.

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